Available online at www.sciencedirect.com

SciVerse ScienceDirect

Procedia - Social and Behavioral Sciences 54 (2012) 808 – 821

Procedia
Social and Behavioral Sciences

15th meeting of the EURO Working Group on Transportation

Traffic equilibrium in a network model of parking and route choice, with search circuits and cruising flows

Fabien Leurent*, Houda Boujnah

Université Paris Est, Laboratoire Ville Mobilité Transport, Ecole des Ponts ParisTech
6-8 avenue Blaise Pascal, 77455 Marne la Vallée Cédex 2, France

Abstract

The paper provides a novel network model of parking and route choice. Supply is represented by parking type, management strategy including the fare, capacity and occupancy rate of parking lot, and network location, in relation to access routes. Demand is addressed in a disaggregate way according to the disposal of private parking facilities and the individual preferences for parking quality of service. Search circuits are explicitly considered on the basis of the success probability to get a slot at a given lot and transition probabilities between lots in case of failure. The basic model variables are the route flows, success probabilities and transition probabilities. These give rise to the cost of a travel route up to a target lot and to the expected cost of search and park from that lot. Each traveller is assumed to make a two stage choice of, first, network route on the basis of the expected overall route costs and, second, local diversion on the basis of a discrete choice model. Traffic equilibrium is defined in a static setting. It is characterized by a mixed problem of variational inequality and fixed point. Equilibrium is shown to exist under mild conditions and a Method of Successive Averages is put forward to solve for it. Lastly, a stylised instance is given to illustrate the effects of insufficient parking capacity on travel costs and network flows.

Keywords: Network assignment, Static assignment, Joint choice, Parking choice. Parking occupancy. Search circuits. Cruising traffic

1. Introduction

Background. Every car trip requires parking the car at the destination place or close to it; it also depends on the parking conditions at the origin place. The parking conditions determine the trip-maker's decisions of travel mode, network route and parking mode, especially so in dense urban areas. It may also involve a specific decision of parking mode in terms of parking type, fare and distance to destination point. Furthermore, the quest for an available parking slot may require specific terminal travel, yielding additional roadway traffic that interferes with the core "through" traffic. Based on 22 US studies, Shoup [1] reports that cruising for parking may represent from

* Corresponding author. Tel.: +33-164-152-111; fax: +33-164-152-140.
E-mail address: fabien.leurent@enpc.fr

 doi:10.1016/j.sbspro.2012.09.797

30 to 50% of road traffic in major city centres, with average search time of 8 minutes for an on-street slot in downtown areas.

Literature review. Models of parking issues may be classified into three categories. First, Discrete Choice Models for the choice of a parking mode emphasize the diversity of modes and management strategies, in relation to the individual behaviour of trip-makers but with little if any consideration of supplied capacity and spatial configuration [2, 3, 4]. Second, parking search and the associated cruising have been modelled as processes of individual behaviour on theoretical grounds and, more recently, on high-resolution network databases through agent-based simulation [5, 6, 7]. Third, parking choices are addressed in conjunction with route choice in the framework of traffic assignment to a network: models are either static [8, 9] or dynamic [10, 11, 12]. However none of them have considered search traffic explicitly, save for [13] in a dynamic setting.

Objective. The paper provides a traffic equilibrium model of parking and route choice on a transportation network including parking facilities. Parking capacity is modelled by lot according to location and management type; its occupancy determines its availability to a candidate user. Search circuits are explicitly considered on the basis of the success probability to get a slot at a given lot and also of transition probabilities to divert to other lots in case of failure. Lot diversion is modelled as a discrete choice on the basis of transition costs and the expected cost of search and parking from the head lot. For simplicity, the setting is static, by assuming that the parking slots are made available in a continuous way due to the departure of their previous occupants. This assumption typically describes the morning peak hour in urban nuclei, when night occupants give place to day occupants motivated notably by work. In the authors' opinion, this makes the major issue of parking in urban transportation planning, since it determines the travel modes chosen in commuting trips.

Approach. Supply is represented by parking type, management strategy including the fare, capacity and occupancy rate of parking lot, and network location, in relation to access routes. Demand is addressed in a disaggregate way according to the disposal of private parking facilities and the individual preferences for parking quality of service. It is assumed that every traveller makes a two-stage choice of, first, network route to a prior target lot on the basis of its expected overall cost and, second, a sequence of local diversions up to parking success. A traffic equilibrium is defined where the individual user selects only a route of minimum expected overall cost to himself. Traffic equilibrium is cast into a joint problem of variational inequality for route choice and fixed point for success as well as transition probabilities. By demand segment, hence destination zone, the expected costs from target lots is evaluated by solving a linear system of small dimension and whose matrix is invertible. It may be thought of as a sophisticated link travel time function, where the "link" refers to a parking lot and involves diversion circuits, while the "travel time" is composed of the parking cost at the final lot plus the circuit cost.

Structure. The rest of the paper is organized in six parts. The assumptions about supply and demand are introduced in Section 2 and 3, respectively. Section 4 brings about a structural analysis of the interaction of supply and demand: the elementary influences are articulated into a logical structure that enables one to identify the core model variables and to state traffic equilibrium as a system of conditions. Then, Section 5 is devoted to more formal though concise mathematical analysis: existence conditions for traffic equilibrium are discussed and a simple computation scheme is put forward. Next, Section 6 deals with a numerical instance to demonstrate parking diversion from more to less demanded lots and the determination of success probabilities. Lastly, Section 7 concludes by pointing to potential developments and on-going work.

2. The supply of parking and route services

On the supply side, transportation services for a motorist include a parking slot as well as a network route, in a joint fashion since the route provides access to the slot which makes the final destination of it.

2.1. Parking supply

Assume that the parking slots are grouped into lots denoted by $\ell \in L$. Each lot is located at a given place and connected to the roadway network. It is operated in a specific mode which imposes a tariff, m_ℓ, and a time of transaction, t_ℓ including cruising and walking time to final destination . During a period of reference, the lot has capacity of κ_ℓ slots which are made available in a progressive way, assumedly by the departure of previous occupants.

Assume also that Y_ℓ customers demand a slot during the period: then the probability of success is

$$\alpha_\ell = \min\{1, \kappa_\ell / Y_\ell\}, \text{ wherein } \alpha_\ell \text{ is set to 1 if } Y_\ell = 0 . \quad (2.1)$$

This probability is a key factor of parking search and associated circuits.

2.2. Network routes

The roadway network is a set A of oriented links a, each one with travel time t_a and money cost m_a per trip to a user during the reference period. Any network route r is a sequence of links $\{a \in r\}$ with continuity of itinerary, route time and money cost as follows:

$$t_r = \sum_{a \in r} t_a , \quad (2.2a)$$

$$m_r = \sum_{a \in r} m_a . \quad (2.2b)$$

Traffic phenomena are modelled by a travel time function with respect to the vector of link flows, $\mathbf{v}_A = [v_a : a \in A]$:

$$\mathbf{t}_A = \mathrm{T}_A(\mathbf{v}_A) . \quad (2.3)$$

3. Demand

3.1. Segments of customers

Demand is analyzed as a set S of segments s of homogeneous customers, each one with a vehicle (presumably a car), an origin zone i_s and a destination zone j_s, a given activity at destination that motivates the trip from i to j and imposes a duration of parking, together with specific economic preferences that include tradeoffs between quality factors such as the time and money costs. The customers within a segment may differ from one another by the disaggregate activity location within the destination zone. Let us assume however that all of them take as valuable a subset L_s of parking lots, located within the destination zone or in the vicinity of it.

3.2. Costs taken for certain

Assume that any customer of segment s evaluates a network route r in a deterministic fashion, synthesized into a single "generalized cost" of travel that depends on the route's money cost and travel time:

$$g_{sr} = \mathrm{G}_s(m_r, t_r) . \quad (3.1)$$

Furthermore, if the customer gets a parking slot in lot ℓ then to him the generalized cost of parking amounts to the following:

$$c_{s\ell} = \mathrm{C}_s(m_\ell, t_\ell) . \quad (3.2)$$

This relationship enables one to account for segment features such as activity duration (hence parking duration) and trade-off between time and money. Later on we shall disaggregate the customers according to activity location and take t_ℓ as variable.

Cost functions G_s and C_s are taken for certain in the sense that they are determined prior to the process of lot search, conditional on the lot that will come out of it.

3.3. Search process and the resulting flows

Each customer participates to the production of his own travel service, by looking for a convenient parking lot. Let us model the related search as the following process:

- The customer selects ex-ante a target lot, say ℓ_0, out of L_s.
- At a current lot ℓ a slot is available with probability α_ℓ. If successful then the customer is satisfied and the search process ends up. Otherwise, the customer directs himself to any lot $n \in L_s$ with diversion probability $\pi^s_{\ell n}$ of transition. This step is repeated until success.

Denote as $\mathbf{p}_s = [\pi^s_{\ell n} : \ell, n \in L_s]$ the matrix of transition probabilities. These depend on, first, the transition due to the operations modes and locations of the two lots, and second on the segment due to economic preferences and the conditions of terminal access (presumably by walk). Any transition $\tau = (\ell, n)$ involves a travel time $t_{s\tau}$ and a money cost $m_{s\tau}$.

Let us analyze the search process by assuming an ex-ante demand of segment s for lots in L_s, $\mathbf{q}_s = [q_{s\ell} : \ell \in L_s]$, and by focusing on the vector of candidate volume by lot, $\mathbf{y}_s = [y_{s\ell} : \ell \in L_s]$.

At any lot ℓ, the number of candidates, $y_{s\ell}$, is made up of the ex-ante selectors, $q_{s\ell}$, plus the candidates diverted from unsuccessful requests, the ℓ-th component of $\mathbf{y}_s \mathbf{J}_{s\alpha} \mathbf{p}_s$, in which $\mathbf{J}_{s\alpha}$ is the diagonal matrix of term $1 - \alpha_\ell$ for $\ell \in L_s$. Thus

$$\mathbf{y}_s = \mathbf{q}_s + \mathbf{y}_s \mathbf{J}_{s\alpha} \mathbf{p}_s . \tag{3.3}$$

or equivalently, denoting by $\mathbf{I}_s$ the identity matrix on L_s,

$$\mathbf{y}_s (\mathbf{I}_s - \mathbf{J}_{s\alpha} \mathbf{p}_s) = \mathbf{q}_s . \tag{3.4}$$

In the appendix, it is shown that matrix $\mathbf{I}_s - \mathbf{J}_{s\alpha} \mathbf{p}_s$ can be inverted; thus, denoting by $\mathbf{H}_{s\alpha p}$ its inverse matrix, it holds that

$$\mathbf{y}_s = \mathbf{q}_s \mathbf{H}_{s\alpha p} . \tag{3.5}$$

Denote by $\tau = (\ell, n)$ the transition from lot ℓ to lot n. To segment s it costs a time of $t_{s\tau}$ and a money expense of $m_{s\tau}$: the two types of costs are aggregated into a "generalized cost" denoted $c_{s\tau}$. The search cost incurred by demand vector $\mathbf{q}_s$ depends on the flow $\mathbf{x}^s_T$ induced on the transitions $\tau \in T_s$ in the following way, denoting by $\vec{\mathbf{p}}_s$ the matrix made up by juxtaposition of square blocks indexed by $\ell \in L_s$, each of which is null save for its ℓ-th row that is taken from $\mathbf{p}_s$:

$$\mathbf{x}_{sT} = \mathbf{y}_s \mathbf{J}_{s\alpha} \vec{\mathbf{p}}_s = \mathbf{q}_s \mathbf{H}_{s\alpha p} \mathbf{J}_{s\alpha} \vec{\mathbf{p}}_s . \tag{3.6}$$

This relationship yields the vector of transition flows, $\mathbf{x}_{sT}$.

3.4. Search cost and the expected costs by target lot

The search cost amounts to the vector product of $\mathbf{x}_{sT}$ times the vector of generalized transition costs, $\mathbf{c}_{sT}$, so

$$\tilde{c}_s(\mathbf{q}_s, \alpha_L, \mathbf{p}_s, \mathbf{c}_{sT}) = \mathbf{x}_{sT} . \mathbf{c}_{sT} = \mathbf{q}_s \, \mathbf{H}_{s\alpha p} \, \mathbf{J}_{s\alpha} \vec{\mathbf{p}}_s . \mathbf{c}_{sT} .$$

A search that starts from lot ℓ corresponds to a particular demand vector $\delta_\ell^s = [1_{\{\ell=n\}} : n \in L_s]$, hence to a particular search cost as follows:

$$\tilde{c}_{s\ell} = \tilde{c}_s(\delta_\ell^s, \alpha_L, \mathbf{p}_s, \mathbf{c}_{sT}) = \delta_\ell^s \, \mathbf{H}_{s\alpha p} \, \mathbf{J}_{s\alpha} \vec{\mathbf{p}}_s . \mathbf{c}_{sT} \; . \tag{3.7}$$

Moreover, the "final" cost of parking also depends on demand vector $\mathbf{q}_s$ through the derived vector $\mathbf{y}_s$:

$$\hat{c}_s(\mathbf{q}_s, \alpha_L, \mathbf{p}_s, \mathbf{c}_{sL}) = \mathbf{y}_s \, \mathrm{diag}(\alpha_L)[c_{s\ell} : \ell \in L_s] = \mathbf{q}_s \, \mathbf{H}_{s\alpha p} \, \mathrm{diag}(\alpha_L) \mathbf{c}_{sL} \, . \tag{3.8}$$

Starting from lot ℓ, the expected cost of search and park is

$$\hat{g}_{s\ell} = \tilde{c}_{s\ell} + \hat{c}_s(\delta_\ell^s, \alpha_L, \mathbf{p}_s, \mathbf{c}_{sL}) = \delta_\ell^s \, \mathbf{H}_{s\alpha p} \, (\mathbf{J}_{s\alpha} \vec{\mathbf{p}}_s \, \mathbf{c}_{sT} + \mathrm{diag}(\alpha_L) \mathbf{c}_{sL}) \, . \tag{3.9}$$

3.5. Travel options and choice behaviour

Ex-post, the travel option of a customer includes a network route r to a target lot ℓ, that lot as ex-ante target and a sequence of transitions starting from ℓ and ending up at first success. The salient features are (r, ℓ) or r only since ℓ must be the final endpoint of r. The expected cost amounts to the travel cost along r, g_{sr}, plus the expected cost from ℓ, $\hat{g}_{s\ell}$:

$$\hat{g}_{sr} = g_{sr} + \hat{g}_{s\ell(r)} \, . \tag{3.10}$$

A two-stage choice behavior is assumed for the customer. First, a travel option of minimum ex-ante cost $\hat{g}_{sr}$ is chosen. Second, depending on the current occupancy state of parking lots, local choice of transition to next lot is made if the customer fails to get a slot at the current lot.

A discrete choice model of behaviour is assumed for local transition choice: local options at ℓ are the transitions starting at current lot and ending at $n \in L_s$, $\tau = (\ell, n)$, with travel disutility of, say, $d_{s\tau} = c_{s\tau} + \hat{g}_{sn} + \varepsilon_{sn}$, in which ε_{sn} is a random variable. The joint distribution of $[\varepsilon_{sn} : n \in L_s]$, together with the transition generalized costs and the lot expected costs, determine the choice probability by transition:

$$\pi_{\ell n}^s = \Pr\{d_{sn} \le d_{sm} : \forall m \in L_s\} \, . \tag{3.11}$$

To sum up, the travel behaviour is rational at both stages on the basis of costs which the customer seeks to minimize. At the network level the cost is an expected cost evaluated in an average, deterministic way (a stochastic part could be added to it to model disaggregate travel conditions at the origin point). At the local level the cost is modelled in a stochastic way – to facilitate the determination of equilibrium and also to address the disaggregate location of destination points.

4. The interaction of supply and demand

The model developed so far consists in a set of variables that are involved in a set of relationships – more precisely, of dependencies. Demand-side variables depend on supply-side variables, and conversely demand variables are factors of supply variables. In this section, our purpose is to gather the dependencies and to lay the emphasis on the overall logical structure.

We shall state each dependency under abstract form, such as $\mathbf{x}_I = \mathbf{X}_I(\mathrm{Z}_J ...)$ in which $\mathbf{x}$ is the dependent variable, I is an index set taken from among routes (R), links (A), lots (L), segments (S) or transitions (T), $\mathbf{Z}_J$ is a factor and $\mathbf{X}_I$ is a mathematical function in abridged notation for an influence that has been stated formerly in a detailed way.

The section is organized in four parts: the three first ones are devoted to gather and re-state the dependencies that pertain to demand (§4.1), traffic (§4.2) and costs (§4.3), respectively. Then the logical structure is depicted in an influence diagram (§4.4).

4.1. Demand functions

At the network level, the assignment of customers to target lots of minimum cost can be stated as follows: Find vector $\mathbf{f}_{SR} = [f_{sr} : r \in R_s, s \in S]$ and dual variables $[\mu_s : s \in S]$ such that

$$f_{sr} \geq 0 \quad \forall r \in R_s,\ \forall s \in S\ , \tag{4.1a}$$

$$\textstyle\sum_{r \in R_s} f_{sr} = Q_s\ , \forall s \in S\ , \tag{4.1b}$$

$$\hat{g}_{sr} - \mu_s \geq 0 \quad \forall r \in R_s,\ \forall s \in S\ , \tag{4.1c}$$

$$f_{sr}(\hat{g}_{sr} - \mu_s) = 0 \quad \forall r \in R_s,\ \forall s \in S\ . \tag{4.1d}$$

At solution, μ_s is equal to the minimum cost among the route options of customer segment s. System (4.1) can be abstracted into a multi-valued mapping as follows:

$$\mathbf{f}_{SR} \in \mathrm{F}_{SR}(\mathbf{Q}_S, \hat{\mathbf{g}}_{SR})\ . \tag{4.2}$$

Lot demand $\mathbf{q}_{SL}$ stems from route flows in a straightforward way: $q_{s\ell} = \sum_{r \in \ell} f_{sr}$, in which { $r \in \ell$ } indicates that lot ℓ is located at the final endpoint of route r. In abstract form,

$$\mathbf{q}_{SL} = \mathrm{Q}_{SL}(\mathbf{f}_{SR})\ . \tag{4.3}$$

Concerning local routing behaviour for the search of an available parking slot, by segment the matrix of transition probabilities, $\mathbf{p}_{ST}$, depends on that of transition costs, $\mathbf{c}_{ST} = [c_{s\tau} : \tau \in T_s, s \in S]$ and on the lot expected costs, $\hat{\mathbf{g}}_{SL} = [\hat{g}_{s\ell} : \ell \in L_s, s \in S]$:

$$\mathbf{p}_{ST} = \mathrm{P}_{ST}(\mathbf{c}_{ST}, \hat{\mathbf{g}}_{SL})\ . \tag{4.4}$$

4.2. Traffic functions

By segment s and parking lot ℓ, the search flow $y_{s\ell}$ depends on the flow inputs of all target lots of that segment, $[q_{sn} : n \in L_s]$, the success probabilities α_L and the transition probabilities $\mathbf{p}_{sL}$: thus

$$\mathbf{y}_{SL} = \mathrm{Y}_{SL}(\mathbf{q}_{SL}, \alpha_L, \mathbf{p}_{ST})\ . \tag{4.5}$$

By segment s and transition $\tau = (\ell, n)$ between lots in L_s, the transition customer volume $x_{s\tau}$ stems from the search flow $y_{s\ell}$, the success probability α_ℓ and the transition probabilities $p^s_{\ell n}$ through $x_{s\tau} = y_{s\ell}.(1-\alpha_\ell).p^s_{\ell n}$. Thus

$$\mathbf{x}_{ST} = \mathrm{X}_{ST}(\mathbf{y}_{SL}, \alpha_L, \mathbf{p}_{ST})\ . \tag{4.6}$$

By demand segment and network link, the route volume due to network routes is derived from the route flows of all segments and network routes: $f_{sa} = \sum_{r \ni a} f_{sr}$, so

$$\mathbf{f}_{SA} = \mathrm{F}_{SA}(\mathbf{f}_{SR})\ . \tag{4.7}$$

At the link level, the flow volume v_{sa} stems from route volumes and also from the network effect of the transition flows. Assuming that a given proportion $\gamma_{as\tau}$ of $x_{s\tau}$ is assigned to link a, then

$$\mathbf{v}_{SA} = \mathrm{V}_{SA}(\mathbf{f}_{SA}, \mathbf{x}_{ST})\ . \tag{4.8}$$

By lot, the probability of parking success, α_ℓ, depends on the lot demand $Y_\ell = \sum_{s \in S} y_{s\ell}$, so

$$\alpha_L = \mathrm{A}_L(\mathbf{y}_{SL}, \kappa_L)\ . \tag{4.9}$$

Link travel times are related to link flows on the basis of travel time functions:

$$\mathbf{t}_{SA} = \mathrm{T}_{SA}(\mathbf{v}_{SA}) . \tag{4.10}$$

In turn, route travel times stem from links' ones by serial composition, $t_{sr} = \sum_{a\in r} t_{sa}$, so

$$\mathbf{t}_{SR} = \mathrm{T}_{SR}(\mathbf{t}_{SA}) . \tag{4.11}$$

4.3. Cost functions

To a customer of segment s , the money cost of travelling once along link a depends on exogenous fares and also on the link travel time (for energy expenses etc), so:

$$\mathbf{m}_{SA} = \mathrm{M}_{SA}(\mathbf{t}_{SA}) . \tag{4.12}$$

The money expenses are composed by route:

$$\mathbf{m}_{SR} = \mathrm{M}_{SR}(\mathbf{m}_{SA}) . \tag{4.13}$$

By route, the generalized cost of travel results from time and money expenses:

$$\mathbf{g}_{SR} = \mathrm{G}_{SR}(\mathbf{t}_{SR}, \mathbf{m}_{SR}) . \tag{4.14}$$

The costs of parking at lots, $\mathbf{c}_{SL}$, are assumed exogenous and may vary by segment. The transition costs depend on the travel times, money expenses, and the linkage $\Gamma_{AST} = [\gamma_{as\tau} : a \in A, s \in S, \tau \in T_s]$ between transitions and network links, on the basis of:

$$\mathbf{c}_{ST} = \mathrm{C}_{ST}(\mathbf{t}_{SA}, \mathbf{m}_{SA}, \Gamma_{AST}) . \tag{4.15}$$

The lot expected costs of search and park comply to

$$\hat{\mathbf{g}}_{SL} = \hat{\mathrm{G}}_{SL}(\alpha_L, \mathbf{p}_{ST}, \mathbf{c}_{ST}, \mathbf{c}_{SL}) . \tag{4.16}$$

Lastly, the route expected cost includes the travel time to target lot plus the search and park cost from that lot:

$$\hat{\mathbf{g}}_{SR} = \hat{\mathrm{G}}_{SR}(\mathbf{g}_{SR}, \hat{\mathbf{g}}_{SL}) . \tag{4.17}$$

4.4. Logical structure

Eqns (4.2)-(4.17) make up an interconnected system of dependencies between the state variables in the model. An overview is provided in Figure 1 in order to depict the logical structure and to trace the various influences. The diagram enables us to identify sub-systems as follows:

- A route demand model in which the route flow vector is determined and yields link flows as well as lot target flows.
- A local demand model for transition probabilities.
- A parking supply and demand model, yielding search flows and success probabilities on the basis of target flows, transition probabilities and lot capacities.
- A traffic model for the determination of transition flows, link flows, link times and route times.
- A costing model to yield link and route money expenses, link / route / lot costs and lot / route expected costs.

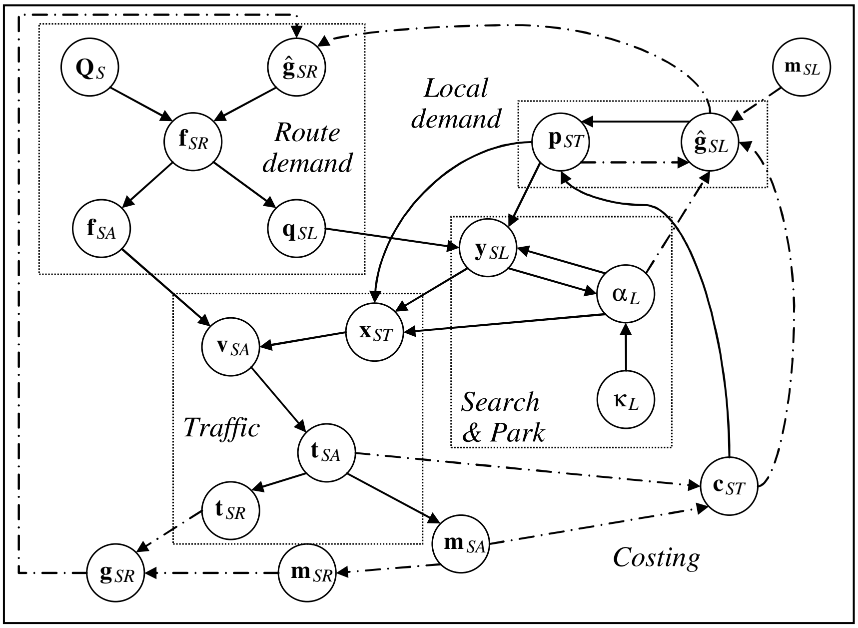

Fig.1. Influences in the routing and parking model

The route demand, traffic and costing models are simple in that their outputs exert no feedback on themselves or the inputs in a straightforward way. But the local demand model and the search and park model exhibit such a straightforward feedback, which makes them harder to solve. Fig. 2 depicts the sub-models as connected by the logical flow.

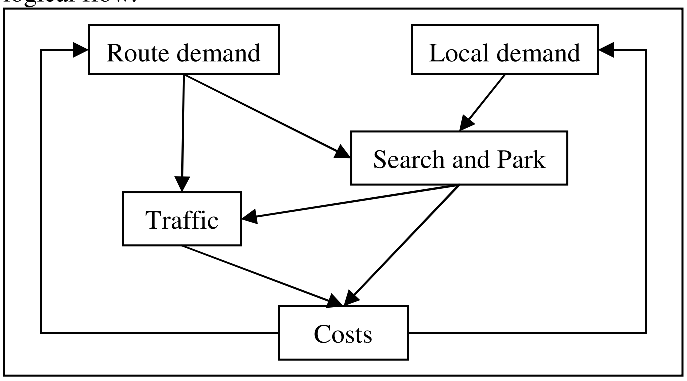

Fig. 2. Sub-models and logical flow

5. Mathematical analysis

Let us now turn to the two major issues of, first, equilibrium i.e. the determination of a system state consistent with all influences and, second, a computation scheme to solve for equilibrium.

5.1. Equilibrium

Most of the variables in the model are endogenous: the exceptions are the segment volumes, the lot capacities and the tariffs by lot or link. Among the endogenous variables, let us select the triple $\mathbf{z} = (\mathbf{f}_{SR}, \mathbf{p}_{ST}, \alpha_L)$ as the "basic" state variable. It can be checked on the influence diagram that, on cutting every arrow which points to any of the three components, then everything else is determined.

Definition: Routing and parking equilibrium. State $\mathbf{z} = (\mathbf{f}_{SR}, \mathbf{p}_{ST}, \alpha_L)$ is an equilibrium if and only if it satisfies jointly the conditions (4.2)-(4.17).

In fact, system (4.2)-(4.17) amounts to a Fixed Point Problem in $\mathbf{z}$: find $\mathbf{z}^*$ such that (4.2), (4.4) and (4.9) hold true when all other endogenous variables are based on $\mathbf{z}^*$.

It is classical to address a route choice model of traffic assignment to a network on the basis of the route flow vector [14]. So it would be tempting to take the same approach to our routing and parking model, by trying to solve in an integrated way for $(\mathbf{p}_{ST}, \mathbf{c}_{ST})$ on the one hand, and for $(\alpha_L, \mathbf{y}_{SL})$ on the other hand. However this would be awkward since the first sub-problem would require much effort, whereas the second problem may have no solution –when $\mathbf{p}_{ST}$ provides little opportunity of lot diversion while the lot target flows are in excess of the local parking capacity.

By replacing system (4.1) of deterministic network routing with analogous conditions of stochastic network routing, then the mapping F_{SR} in (4.2) would be a continuous function and the system (4.2)-(4.17) would make a continuous function of $\mathbf{z}$. As $\mathbf{p}_{ST}$ and α_L are probabilities, and the route flows are non negative and bounded by the segment volume, the admissible set is compact. Let us assume that the system is feasible, i.e. that the overall parking capacity is greater than the overall trip volume and that the subsets L_s enable for sufficiently wide dispersion of local demand. Then the admissible set is compact and nonempty so that, if the fixed point function is continuous, there must be a solution of routing and parking equilibrium.

This proof can be extended to deterministic routing by taking it as the limiting case of stochastic routing. The uniqueness of an R&P equilibrium is still an open issue – at least concerning the success probabilities.

5.2. Computation scheme

The R&P equilibrium can be searched for by solving a mixed problem of variational inequality (on route flows) and fixed point (on success probabilities and transition probabilities). As a first, simple approach, let us address the variational inequality by a Method of Successive Averages, i.e. convex combination of current state $\mathbf{f}_{SR}^{(k)}$ with an auxiliary state $\tilde{\mathbf{f}}_{SR}^{(k)}$ that solves (4.1) with respect to the current state of costs, into a new state $\mathbf{f}_{SR}^{(k+1)} = \mathbf{f}_{SR}^{(k)} + \zeta_k (\tilde{\mathbf{f}}_{SR}^{(k)} - \mathbf{f}_{SR}^{(k)})$ where ζ_k is a decreasing sequence of numbers in]0,1[save for $\zeta_0 = 1$.

On the probability side, the new state can be obtained by convex combination also but using a fixed coefficient, say ω_α for success probabilities and ω_p for transition probabilities.

Here is an abridged flowchart:

- *Initialization*. Let $k := 0$, $\alpha_\ell := 1\ \ \forall \ell \in L$, $p^s_{\ell\tau} := 1/|L_s|\ \ \forall \ell, n \in L_s$ and $\forall s \in S$, $\mathbf{f}^{(0)}_{SR} := \mathbf{0}$.
- *Route and lot costing*. Based on $\mathbf{z}^{(k)}$, evaluate link costs, route costs, transition costs, lot expected costs and route expected costs, $\hat{\mathbf{g}}^{(k)}_{SR}$.
- *Auxiliary state*. Assign demand volumes to routes of minimum expected cost, yielding $\tilde{\mathbf{f}}^{(k)}_{SR}$. From these and the current probabilities, derive the $\tilde{\mathbf{q}}^{(k)}_{SL}$ and $\tilde{\mathbf{y}}^{(k)}_{SL}$. Then derive $\tilde{\alpha}^{(k)}_L$ on the basis of (4.9) and $\tilde{\mathbf{p}}^{(k)}_{ST}$ on the basis of the current transition costs and expected lot costs.
- *Convex combination*. Let $\mathbf{f}^{(k+1)}_{SR} := \mathbf{f}^{(k)}_{SR} + \zeta_k(\tilde{\mathbf{f}}^{(k)}_{SR} - \mathbf{f}^{(k)}_{SR})$, $\alpha^{(k+1)}_L := \alpha^{(k)}_L + \omega_\alpha(\tilde{\alpha}^{(k)}_L - \alpha^{(k)}_L)$ and $\mathbf{p}^{(k+1)}_{ST} := \mathbf{p}^{(k)}_{ST} + \omega_p(\tilde{\mathbf{p}}^{(k)}_{ST} - \mathbf{p}^{(k)}_{ST})$.

• *Convergence test*. If distance between $\mathbf{z}^{(k)}$ and $\mathbf{z}^{(k+1)}$ is small enough then stop, else increment k and go to Costing step.

As a distance criterion, a sum of functions by component in $\mathbf{z}$ is appropriate: for instance a duality gap on $\mathbf{f}_{SR}$ and squared distances on each probability vector, with formulae as follows:

$$\mathrm{DG}(\mathbf{f}_{SR}) = \textstyle\sum_{s,r} \hat{\mathbf{g}}^{(k)}_{sr}(\mathbf{f}^{(k)}_{sr} - \tilde{\mathbf{f}}^{(k)}_{sr}),$$
$$\mathrm{D}^2_\alpha = \left\|\tilde{\alpha}^{(k)}_L - \alpha^{(k)}_L\right\|^2 \text{ and } \mathrm{D}^2_p = \left\|\tilde{\mathbf{p}}^{(k)}_{ST} - \mathbf{p}^{(k)}_{ST}\right\|^2.$$

6. Stylized instance

6.1. Case design

Let us consider an urban nucleus, with a dense central area also called Ring 0, and peripheral areas more or less close to the centre, whence grouped into either the first or the second ring. Each peripheral ring is divided into N areas. Each area is a destination zone with only one parking lot located at its centroid. So there are three types of lots, denoted by $\ell \in \{0,1,2\}$. Radial symmetry is assumed, see Fig. 3. Lot capacities κ_ℓ yield a total parking capacity, $\mathrm{K} = \kappa_0 + N\kappa_1 + N\kappa_2$. The network links connect neighbouring lots only.

On the demand side, there is one demand segment by destination zone. For simplicity, route choice is given exogenously so that only the local centroid is taken as ex-ante target lot. Routing on the main network is neglected so that the origin zones are unimportant. Furthermore, local routing also is exogenous. Centre-destined trips, i.e. Segment 0, want to park in Ring 0 or 1: from lot 0 the diversion probability to each lot in the first ring must be $p^{(0)}_{0n} = 1/N$; from any lot in the first ring the diversion probability to a neighbouring lot in the same ring is $p^{(0)}_{n,n+\varepsilon} = \beta$ for $\varepsilon = \pm 1$ and 0 otherwise, and $p^{(0)}_{n0} = 1 - 2\beta$. It is shown in the Appendix that demand Q_0 in segment 0 yields search flows of $y^{(0)}_0 = Q_0(1 - 2\beta\overline{\alpha}_1)/\sigma$ at lot 0 and $y^{(0)}_{1n} = Q_0\overline{\alpha}_0/[N\sigma]$ at any lot in ring 1, where $\overline{\alpha}_\ell = 1 - \alpha_\ell$ and $\sigma = 1 - 2\beta\overline{\alpha}_1 - \overline{\alpha}_0\overline{\alpha}_1(1 - 2\beta)$.

Every destination in the first ring has its own segment with demand flow Q_1 targeted at its centroid. Only this and the five neighbouring lots are taken as attractive, yielding a system of transitions depicted in Fig. 3c. By symmetry, the 20 transitions involve only 10 parameters.

Lastly, any destination belonging to the second ring has demand flow Q_2 which meets sufficient capacity at its initial target, so that $\alpha_{2n} = 1$ and no trip needs to divert to another lot. This implies also that no diversion occur from nodes d^+ and d^- of segment 1: hence the six transitions coming out of these nodes can be cut off.

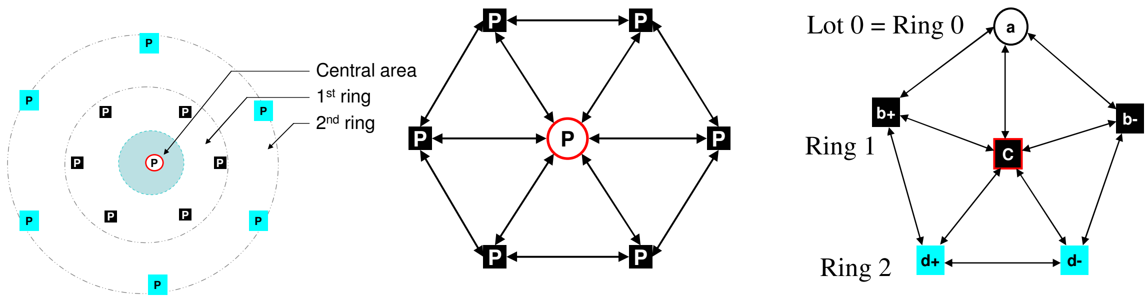

Fig. 3. Rings of parking lots (a); Transition network of Segment 0 (b) and Segment 1n (c)

6.2. Parking equilibrium and parametric analysis

Routing has been much simplified so as to focus on parking loading rate under initial demand, say $Q_{\ell n} / \kappa_{\ell n}$, success rate $\alpha_{\ell n}$ and looping ratio $y_{s\ell} / q_\ell$. The basic endogenous variables are reduced to α_0 and α_1 since $\alpha_2 = 1$. Vector α_L combined with exogenous ex-ante flows $\mathbf{Q}_L$ and diversion probabilities $\mathbf{p}_{ST}$ yields search flows $\mathbf{y}_{SL}$ and, in turn, lot demand $\mathbf{Y}_L$ as follows: by symmetry, $Y_0 = y_0^{(0)} + Ny_a^{(1)}$ while $Y_{1,n} = y_1^{(0)} + y_c^{(1)} + 2y_b^{(1)}$ and $Y_{2,n} = 2y_d^{(1)} + Q_2$. These demands are faced to lot capacities and determine the success rates, α_L. Traffic equilibrium reduces to a fixed point problem in two variables, α_0 and α_1.

Table 1 indicates the parameter setting apart from $N = 6$. Fig. 4 to 6 depict the main model outcomes: success rates by lot, search volume by type of transition link, expected cost of target lot by demand segment, with respect to parameter Q_0 i.e. the initial inflow of parking demand in the central area. Value 2,000 of Q_0 would saturate the overall parking capacity. As Q_0 is increased towards this bound, the success probabilities decrease for the diversion lots of Segment 0 i.e. those in Rings 0 and 1 (figure 4). The search flows of Segment 0 keep increasing with a hyperbolic trend (figure 5). The same applies to the search cost per trip in Segment 0, while that in Segment 1 increases linearly (figure 6). In practice, very high search costs would drive the trip-makers in Segment 0 to target lots in Ring 1 in their ex-ante route choice, thus reducing their search cost to that of Segment 1; similarly, local diversion would favour lots in Ring 1 rather than the central one; lastly, the trip-makers in Segment 0 could divert to farther lots and try to park in Ring 2 as well as in Rings 0 and 1.

Table 1. Parameter setting

Lot capacity (veh/h)	Segment inflow (veh/h)	Transition probability	Transition cost
$\kappa_0 = 500$	Q_0 variable	Segment 0: $\beta = 0.4$	Set to 1 whatever the link
$\kappa_{1n} = 250$	$Q_{1n} = 500$	Segment 1:	
$\kappa_{2n} = 1{,}000$	$Q_{2n} = 500$	$p_{ca} = p_{cb} = p_{cd} = 0.2$	
		$p_{ac} = .6, p_{ab} = 0.2$	
		$p_{bc} = .3, p_{ba} = 0.2, p_{bd} = 0.5$	

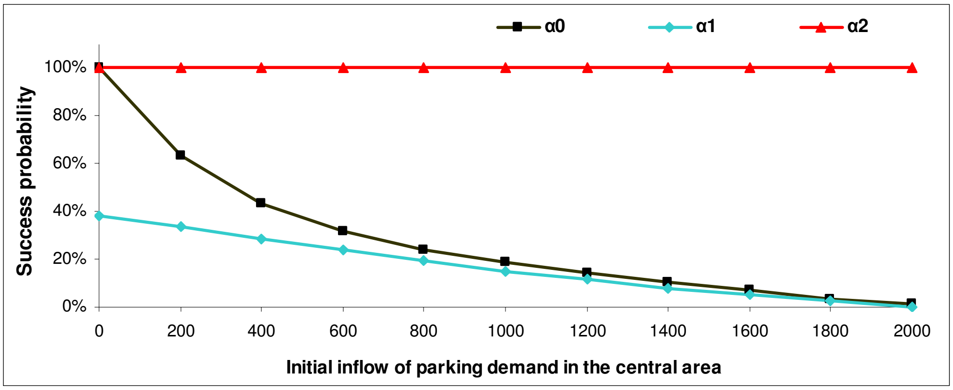

Fig. 4. Success probabilities

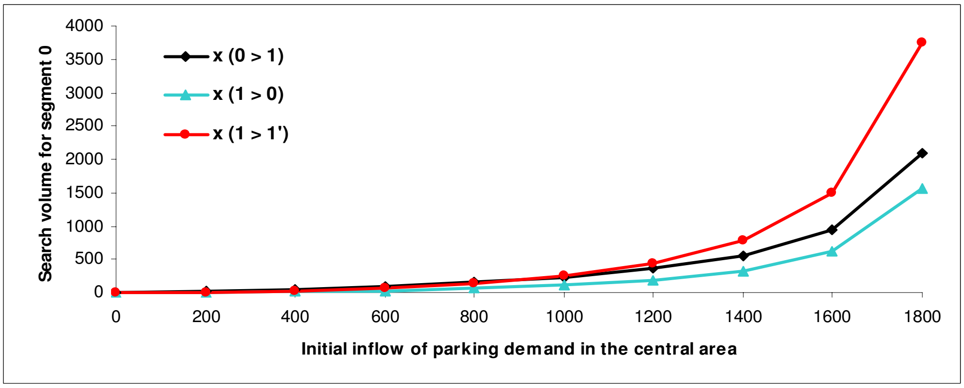

Fig. 5. Search flow of Segment 0, by link type

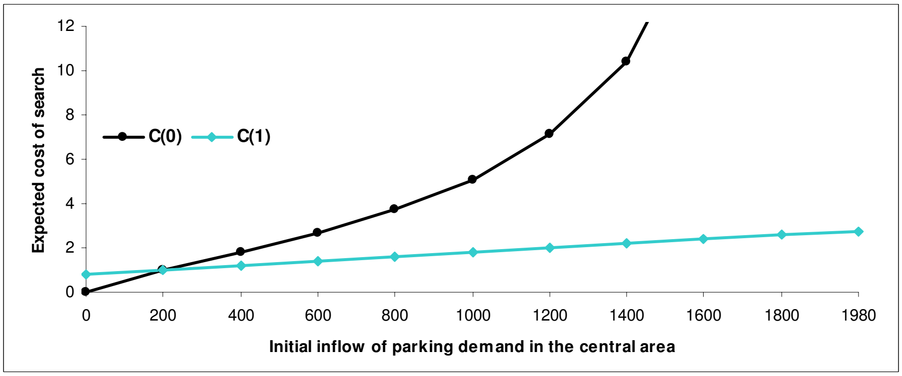

Fig. 6. Search cost per trip according to Segment

7. Conclusion

A joint model of routing and parking along a roadway network has been provided, in which microeconomic features of individual behaviour are integrated to macroscopic features of parking occupancy and link flows. Cruising traffic is explicit and interacts with the flows of network routing. The static setting allows for simple application to a prominent issue in urban transportation planning: that of capacity planning for roadways and parking lots.

The model may be developed in two directions. On the supply side, it is tempting to distinguish time periods within day so as to capture the variation of lot occupancy in response to the location in time and space of activity purposes: the model of [13] could be simplified by identifying two time scales and restricting the search circuits to the smaller one, in a quasi-static way. On the demand side, the features of network route and parking lot interact in reality with the choices of travel mode and departure time: search circuits could be integrated into the model of, e.g., [12].

Acknowledgements

This research was supported by the Research and Education Chair on “Eco-design of buildings and infrastructure”, operated by Mines, Agro and Ecole des Ponts ParisTech in partnership with the Vinci Group, to which we are grateful.

References

[1] Shoup D.C,.(2006), Cruising for parking, Transport Policy, N°13, pp 479-486.
[2] Polak J.W., Axhausen K. (1990). Parking Search Behaviour: A review of current Research and Future prospects, Transportation Studies Unit, Working paper 540, Oxford University.
[3] Hensher D.A., King J. (2001). Parking demand and responsiveness to supply, pricing and location in the Sydney central business district," Transportation Research Part A,. 35/3: 177-196.
[4] Hess S., Polak J.W. (2004). Mixed logit estimation of parking type choice, Presented at the 83rd annual meeting of the Transportation Research Board, Washington D.C.
[5] Benenson I., Martens K., Birfir S. (2008). PARKAGENT: An agent-based model of parking in the city, Computers, Environment and Urban Systems 32: 431–439.
[6] Dieussaert K., Aerts K., Steenberghen T., Maerivoet S., Spitaels K. (2009). SUSTAPARK: An Agent-based Model for Simulating Parking Search, Proceedings of the 12th AGILE International Conference on Geographic Information Science, Leibnitz Universität, Hannover, Germany, June 2009.
[7] Martens C.J.C.M., Benenson I., Levy N. (2010). The Dilemma of On-Street Parking Policy: Exploring Cruising for Parking Using an Agent-Based Model, in Bin, J.;Xiaobai Y. (eds), Geospatial Analysis and Modelling of Urban Structure and Dynamics, Heidelberg, DE : Springer Verlag.
[8] Gur Y.J., Beimborn E.A. (1984). Analysis of Parking in Urban Centers: Equilibrium Assignment Approach, Transportation Research Record 957: 55-62.
[9a] Li Z.C., Huang H..J.., Lam WHK, Wong S.C. (2007). A model for evaluation of transport policies in multimodal networks with road and parking capacity constraints, Journal of Mathematical Modelling and Algorithms, 6, pp.239-257.
[9b] Li Z.C, Lam W.H.K., Wong S.C., Huang H.J., Zhu D.L. (2008). Reliability evaluation for stochastic and time-dependent networks with multiple parking facilities, Networks and Spatial Economics, 8:55-381.
[10] Bifulco G.N. (1993). A stochastic user equilibrium assignment model for the evaluation of parking policies, European Journal of Operational research 71/2: 269-287.
[11] Lam W.H.K., Li Z.C., Huang H.J., Wong S.C. (2006). Modeling time-dependent travel choice problems in road networks with multiple user classes and multiple parking facilities, Transp Res 40B: 368–395
[12] Li Z.C., Huang H.J., Lam W.H.K., Wong S.C. (2007). Time-differential pricing of road tolls and parking charges in a transport network with elastic demand. In: Bell MGH et al. (eds) Transportation and traffic theory.

[13] Gallo M., D'Acierno L., Montella B. (2011). A multilayer model to simulate cruising for parking in urban areas, Transport Policy, Elsevier, 18/5: 735-744.
[14] Sheffi Y. (1985). Urban transportation networks: Equilibrium Analysis with Mathematical Programming Methods, Prentice Hall. Englewood Cliff, New Jersy.
[15] Leurent F., Boujnah H. (2011). Une analyse offre-demande du stationnement. Application à l'agglomération parisienne, In : Proceedings of the ATEC-ITS congress, Versailles, France, 2-3 February.

Appendix A

A.1. Invertibility of matrix $\mathbf{I}_s - \mathbf{J}_{s\alpha}\mathbf{p}_s$

Recall that $\mathbf{p}_s$ is a matrix of transition probabilities while $\mathbf{J}_{s\alpha}$ is the diagonal matrix of failure probabilities, $\overline{\alpha}_\ell = 1 - \alpha_\ell$ for $\ell \in L_s$. Assume that $\alpha_\ell > 0$ i.e. that every lot has positive capacity. Then $\overline{\alpha}_\ell < 1$. Consider the series of matrices, $\mathbf{M}^n = \sum_{i=0}^{n} (\mathbf{J}_{s\alpha}\mathbf{p}_s)^i$.

As matrix $\mathbf{J}_{s\alpha}$ is diagonal with coefficients in [0,1[and $\mathbf{p}_s$ is a probability matrix, the product $\mathbf{J}_{s\alpha}\mathbf{p}_s$ has modulus strictly less than one, hence $\mathbf{M}^n$ converges to a given matrix $\mathbf{M}^*$ as n tends to infinity. It holds that

$(\mathbf{I}_s - \mathbf{J}_{s\alpha}\mathbf{p}_s)\mathbf{M}^n = \sum_{i=0}^{n} (\mathbf{J}_{s\alpha}\mathbf{p}_s)^i - \sum_{i=0}^{n} (\mathbf{J}_{s\alpha}\mathbf{p}_s)^{i+1} = \mathbf{I}_s - (\mathbf{J}_{s\alpha}\mathbf{p}_s)^{n+1}$, so that

$(\mathbf{I}_s - \mathbf{J}_{s\alpha}\mathbf{p}_s)\mathbf{M}^* = \mathbf{I}_s$ which shows that $\mathbf{M}^*$ is the inverse matrix of $\mathbf{I}_s - \mathbf{J}_{s\alpha}\mathbf{p}_s$.

A.2. Analytical solution of stylised instance

Denote $\mathbf{y}(\mathbf{I} - \mathbf{Jp}) = \mathbf{q}$ the problem of candidate flows at lot 0. Matrix $\mathbf{I} - \mathbf{Jp}$ is patterned as follows:

$$\begin{bmatrix} 1 & -a & -a & -a & \dots & -a & -a \\ -c & 1 & -b & 0 & \dots & 0 & -b \\ -c & -b & 1 & -b & \dots & \dots & 0 \\ -c & 0 & -b & 1 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ -c & 0 & \dots & 0 & \dots & 1 & -b \\ -c & -b & 0 & \dots & 0 & -b & 1 \end{bmatrix} \quad \text{where} \quad \begin{cases} a = \overline{\alpha}_0 / N \\ b = \overline{\alpha}_1 \beta \\ c = \overline{\alpha}_1 (1 - 2\beta) \end{cases}$$

The first two rows in the problem yield equations

$$y_0 - c\sum_{n=1}^{N} y_{1n} = q_0$$
$$-ay_0 + y_{1n} - by_{1,n-1} - by_{1,n+1} = q_{1n}$$

By symmetry, $y_{1n} = y_1, \forall n$, yielding that

$$y_0 - cNy_1 = q_0$$
$$(1 - 2b)y_1 = q_1 + ay_0$$

Denote $\sigma = 1 - 2b - acN = 1 - 2\overline{\alpha}_1\beta - \overline{\alpha}_0\overline{\alpha}_1(1 - 2\beta)$. It comes out that

$$y_1 = \frac{a}{\sigma} q_0 + \frac{1}{\sigma} q_1$$
$$y_0 = \frac{1-2b}{\sigma} q_0 + \frac{cN}{\sigma} q_1$$